VISCOUS LIQUID FLOW IN A CHANNEL WITH SINUSOIDAL WALLS

V. I. Borisov

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An approximate solution is obtained for the problem of flow of a viscous incompressible liquid within a plane channel formed by uniformly spaced sinusoidally curved walls.

The present study will consider the flow of a viscous incompressible liquid in a plane channel with sinusoidally curved walls. The distance between the walls is constant and the flow is assumed steady-state. Such a flow is realized in a gas–liquid separator, and the results obtained can be used to calculate the process of droplet seeding on the separator walls.

A number of studies exist [1-4], dedicated to flow of viscous liquids in curved channels. The majority of these deal with the axisymmetric case. In [4] the channel geometry coincides with that considered here, but the equations of motion were used in the Stokes approximation. The solution presented below is based on the complete Navier–Stokes equations.

We will now formulate the problem. The equation describing liquid flow has the form [5]

$$\frac{\partial\Psi}{\partial Y}\frac{\partial\Delta\Psi}{\partial X}-\frac{\partial\Psi}{\partial X}\frac{\partial\Delta\Psi}{\partial Y}=\nu\Delta\Delta\Psi,\quad \Psi=\Psi(X,\ Y). \tag{1}$$

The boundary conditions are the condition of liquid adhesion to the walls, and the constancy of fluid flow Q at any channel section:

$$Y=-h+a\cos\left(\frac{2\pi}{\lambda}X\right),\quad \Psi=0,\quad \frac{\partial\Psi}{\partial Y}=0, \tag{2}$$

$$Y=h+a\cos\left(\frac{2\pi}{\lambda}X\right),\quad \Psi=Q,\quad \frac{\partial\Psi}{\partial Y}=0. \tag{3}$$

The channel form, defined by three geometric parameters, the half-width h, the amplitude a, and the wavelength λ, is shown in Fig. 1.

With the introduction of dimensionless variables $x'=X/h$, $y'=Y/h$, $\psi(x', y')=\Psi/\nu$, Eqs. (1)-(3) take on the form:

$$\frac{\partial\psi}{\partial y'}\frac{\partial\Delta\psi}{\partial x'}-\frac{\partial\psi}{\partial x'}\frac{\partial\Delta\psi}{\partial y'}=\Delta\Delta\psi, \tag{4}$$

$$y'=-1+\frac{a}{h}\cos\left(\frac{2\pi h}{\lambda}x'\right),\quad \psi=0,\quad \frac{\partial\psi}{\partial y'}=0, \tag{5}$$

$$y'=1+\frac{a}{h}\cos\left(\frac{2\pi h}{\lambda}x'\right),\quad \psi=\frac{Q}{\nu},\quad \frac{\partial\psi}{\partial y'}=0. \tag{6}$$

We now introduce the notation $A=a/h$, $\overline{Q}=Q/\nu$, $\varepsilon=2\pi h/\lambda$, and assume that ε is a small quantity. To solve Eqs. (4)-(6) we use the Blasius method [6], developed for the problem of liquid flow in a tube with slowly varying cross section.

We perform the substitution $x=\varepsilon x'$, $y=y'$, eliminating ε from the boundary conditions and introducing it into the equation. Instead of Eqs. (4)-(6) we now have

$$\varepsilon^3(\psi_y\psi_{xxx}-\psi_x\psi_{xxy})+\varepsilon(\psi_y\psi_{yyx}-\psi_x\psi_{yyy})=\varepsilon^4\psi_{xxxx}+\varepsilon^2 2\psi_{xxyy}+\psi_{yyyy}, \tag{7}$$

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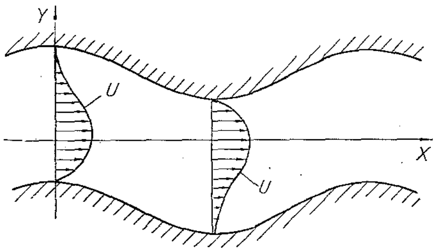

Fig. 1. Channel form and longitudinal velocity profile at two sections.

$$y = -1 + z(x), \quad \psi = 0, \quad \psi_y = 0, \tag{8}$$

$$y = 1 + z(x), \quad \psi = \overline{Q}, \quad \psi_y = 0. \tag{9}$$

Here and below partial derivatives of the function ψ with respect to the variables x and y are denoted by corresponding subscripts. In addition we use the notation $z(x) = A \cos x$.

We will seek a solution of Eqs. (7)-(9) in the form

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots \tag{10}$$

Substituting Eq. (10) in system (7)-(9), we sequentially obtain zeroth, first, second, etc. approximations.

The zeroth approximation:

$$\psi_{0yyyy} = 0, \quad y = -1 + z, \quad \psi_0 = 0, \quad \psi_{0y} = 0, \quad y = 1 + z,$$
$$\psi_0 = \overline{Q}, \quad \psi_{0y} = 0.$$

The solution of the zeroth approximation is a Poiseuille flow:

$$\psi_0 = \frac{\overline{Q}}{4}(-\eta^3 + 3\eta + 2), \tag{11}$$

where $\eta = y - z(x)$.

The first approximation:

$$\psi_{1yyyy} = 0, \quad y = -1 + z, \quad \psi_1 = 0, \quad \psi_{1y} = 0, \quad y = 1 + z,$$
$$\psi_1 = 0, \quad \psi_{1y} = 0.$$

The solution is $\psi_1 = 0$.

The second approximation:

$$\psi_{2yyyy} = -3\overline{Q}z'', \quad y = -1 + z, \quad \psi_2 = 0, \quad \psi_{2y} = 0, \quad y = 1 + z,$$
$$\psi_2 = 0, \quad \psi_{2y} = 0.$$

The solution is

$$\psi_2 = -\frac{\overline{Q}z''}{8}(\eta^2 - 1)^2. \tag{12}$$

Here the primes denote derivatives of the function $z(x)$.

The third approximation:

$$\psi_{3yyyy} = \frac{3\overline{Q}^2}{8} z'''(\eta^4 - 1) + \frac{9\overline{Q}^2}{4} z'z''\eta(\eta^2 - 1),$$
$$y = -1 + z, \quad \psi_3 = 0, \quad \psi_{3y} = 0, \quad y = 1 + z, \quad \psi_3 = 0, \quad \psi_{3y} = 0.$$

The solution is

$$\psi_3 = \frac{3}{8}\overline{Q}^2\left[\frac{z'''}{1680}(\eta^2 - 1)^2(\eta^4 + 2\eta^2 - 67) + \frac{z'z''}{140}\eta(\eta^2 - 1)^2(\eta^2 - 5)\right]. \tag{13}$$

Substituting Eqs. (11)-(13) in Eq. (10), we obtain the following expression for the flow function:

$$\psi = \frac{\overline{Q}}{4}(-\eta^3 + 3\eta + 2) - \varepsilon^2\frac{\overline{Q}z''}{8}(\eta^2 - 1) + \varepsilon^3\frac{3}{8}\overline{Q}^2\left[\frac{z'''}{1680}(\eta^2 - 1)^2(\eta^4 + 2\eta^2 - 67) + \frac{z'z''}{140}\eta(\eta^2 - 1)^2(\eta^2 - 5)\right]. \tag{14}$$

Mathematically, Eq. (14) consists of the first three terms of an asymptotic expansion of the desired solution as $\varepsilon \to 0$ and arbitrary $\bar{Q}$ = const and A = const (the parameter $A = a/h$ appears in $z = A \cos x$). In order to use the solution in practice, it is necessary to select the parameters ε, $\bar{Q}$, and A such that the first term dropped be much smaller than the preceding one. Then, as is evident from Eq. (14), $\bar{Q}$ and A cannot be too large. Since the dimensionless flow rate $\bar{Q} = Q/\nu = U_m \cdot 2h/\nu$ is the Reynolds number constructed with the mean velocity U_m and channel width, it can be said that the solution obtained is limited to Reynolds numbers which are not too large.

We will now use the solution obtained to determine the conditions for development of breakoff of the flow in the curved channel. To do this we use the first two terms of Eq. (14). Then the horizontal (dimensionless) flow velocity is defined by the expression

$$u = \frac{\partial \psi}{\partial y} = \frac{3\bar{Q}}{4}(1-\eta^2) + \varepsilon^2 \frac{\bar{Q}z''}{2}\eta(1-\eta^2). \tag{15}$$

The condition for flow breakoff on the wall is

$$\frac{\partial u}{\partial y} = 0. \tag{16}$$

Substituting Eq. (15) in Eq. (16) and considering, for concreteness, the lower wall (i.e., $\eta = -1$), we obtain $2\varepsilon^2 z'' = 3$; But $z'' = -A \cos x$, and the breakoff condition becomes

$$-2\varepsilon^2 A \cos x = 3. \tag{17}$$

It follows from Eq. (17) that breakoff on the lower wall can develop only at those points where $\cos x < 0$, while the points most dangerous in this sense are $x = \pi(2k+1)$, $k = 0, \pm1, \pm2, \ldots$, i.e., the concavities of the channel profile.* The condition for breakoff at such a point will be $\varepsilon^2 A = 3/2$, or in dimensional form

$$\frac{ah}{\lambda^2} = \frac{3}{8\pi^2}. \tag{18}$$

A similar condition is true for breakoff on the upper channel wall, where the peaks of the profile are the most dangerous points.

It is interesting that breakoff condition (18) contains only the geometric characteristics of the channel, and is independent of both flow rate and liquid viscosity.

NOTATION

a, sinusoid amplitude; A, dimensionless amplitude; h, channel half-width; k, integer; Q, volume flow rate; $\bar{Q}$, dimensionless flow rate; U, longitudinal velocity; U_m, mean longitudinal velocity; u, dimensionless longitudinal velocity; X, longitudinal coordinate; x', x, dimensionless longitudinal coordinates; Y, transverse coordinate; y', y, dimensionless transverse coordinates; Δ, Laplacian; ε, expansion parameter; λ, sinusoid wavelength; η, dimensionless coordinate; ν, kinematic viscosity; Ψ, flow function; ψ, dimensionless flow function.

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*We note that at these points the term corresponding to the third approximation is identically equal to zero.